RA 16

8.15 Solving linear homogeneous recurrence relations

- Finding an explicit formula for a recursively defined sequence is called solving a recurrence relation.
- Each number in a sequence defined by linear homogeneous recurrence relation is a linear combination of numbers that occur earlier in the sequence.
- In a homogeneous recurrence relation, the expression for S_n contains no additional terms besides the ones that refer to earlier numbers in the sequence.

Examples: (circled part is what makes it non-linear or nonhomogeneous)

Recurrence Relation	Type
$b_n = b_{n-1} + (b_{n-2} \cdot b_{n-3})$	Nonlinear
$c_n = 3n \cdot c_{n-1}$	Nonlinear
$d_n = d_{n-1} + (d_{n-2})^2$	Nonlinear
$f_n = 3f_{n-1} - 2f_{n-2} + f_{n-4} + n^3$	Linear, degree 4, Nonhomogeneous
$g_n = g_{n-1} + g_{n-3} + 1$	Linear, degree 3, Nonhomogeneous
$h_n = 2h_{n-1} - h_{n-2}$	Linear, degree 2, Homogeneous
$s_n = 2s_{n-1} - \sqrt{3}(s_{n-5})$	Linear, degree 5, Homogeneous

- If the number x satisfies the characteristic equation for a linear homogeneous recurrence relation, then x^n satisfies the recurrence relation.

$\boxed{\text{Example}}$ Recurrance Relation: $f_n = f_{n-1} + f_{n-2}$

Assume: $f_n = x^n$, $f_{n-1} = x^{n-1}$, $f_{n-2} = x^{n-2}$

Then: $x^n = x^{n-1} + x^{n-2}$

$$x^2 \cdot x^{n-2} = x \cdot x^{n-2} + 1 \cdot x^{n-2}$$

$$x^2 = x + 1$$

$$\boxed{x^2 - x - 1 = 0} \leftarrow \text{characteristic equation}$$

- Once the characteristic equation has been determined, the next step is to find all values of x that solve the equation.

- Then any linear combination of the solutions to the characteristic equation is the <u>general solution</u> to the recurrence relation.

$\boxed{\text{Ex}}$ $x_1 = 2$ $x_3 = 4$

general solution: $a(2) + b(4)$, $a, b \in \mathbb{R}$

- The final step is to find a,b such that initial condition is satisfied.

General Steps to solve with an example:

$$f_0 = 7, \quad f_1 = 0, \quad f_2 = 10, \quad f_n = 2(f_{n-1}) + (f_{n-2}) - 2(f_{n-3})$$

General Step	Example
• Use recurrence relation to find the characteristic equation which is $p(x) = 0$, where $p(x)$ is a degree d polynomial.	$x^3 - 2x^2 - x + 2 = 0$
• Find all d solutions to the characteristic equation.	$x^3 - 2x^2 - x + 2 = (x-2)(x+1)(x-1) = 0$ $x_1 = 2, \ x_2 = 1, \ x_3 = -1$
• Every solution $f_n (x_1)^n$ satisfies the recurrence relation. Therefore, any solution of the form $f_n = a_1 (x_1)^n + \dots + a_d (x_d)^n$ satisfies the relation.	$f_n = a_1 2^n + a_2 (1)^n + a_3 (-1)^n$
• Each initial value gives a value of f_n for a specific n. Use this to set up a system.	$n=0: f_0 = 7 = a_1 + a_2 + a_3$ $n=1: f_1 = 0 = 2a_1 + a_2 - a_3$ $n=2: f_2 = 10 = 4a_1 + a_2 + a_3$
• Solve for $a_1, \dots, a_d$	$a_1 = 1, \ a_2 = 2, \ a_3 = 4$
• Plug values for $a_1, \dots, a_d$ back into the expression for f_n to get the closed form expression for f_n.	$f_n = 1(2^n) + 2(1^n) + 4(-1^n)$ $= 2^n + 2 + 4(-1)^n$

Example (matching):

Characteristic Equation	Solutions to Recurrence Relation
$(x+1)(x-3)(x-4)(x+2)$	$a_1(-1)^n + a_2(3)^n + a_3(4)^n + a_4(-2)^n$
$(x-3)^4$	$a_1(3)^n + a_2(n)(3)^n + a_3(n^2)3^n + a_4(n^3)3^n$
$(x+1)(x-3)(x-2)^2$	$a_1(-1)^n + a_2 3^n + a_3 2^n + a_4(n)2^n$

8.16 Solving linear nonhomogeneous recurrence relations

Nonhomogeneous RR	Associated homogeneous RR
$f_n = 2f_{n-1} + 12f_{n-3} + 3 \cdot 7^n + 21$	$f_n = 2f_{n-1} + 12f_{n-3}$
$g_n = 3g_{n-2} + \sqrt{n} + \log n$	$g_n = 3g_{n-2}$

- The solution of non-homogeneous linear RR is the sum of a homogeneous solution $(f_n^{(h)})$ and the particular solution $(f_n^{(p)})$.

Summary of Steps:

- The example is: $f_n = 3f_{n-1} + 10f_{n-2} + 24n$, $f_0 = \frac{7}{6}$, $f_1 = -\frac{11}{6}$

General Step	Example
• Find the homogeneous part of the solution, which is the general solution to the associated homogeneous RR.	• $f_n^{(h)} = a_1 5^n + a_2(-2)^n$ which is general solution to: $f_n = 3f_{n-1} + 10f_{n-2}$
• Guess the correct form for the particular solution.	• Guess: $f_n^{(p)} = an+b$ for some constants a and b
• Verify the guess by solving for constants so that the guess satisfies the RR for all n.	• Find constants a and b so that $an+b = 3[a(n-1)+b] + 10[a(n-2)+b] + 24n$ for every n $a = -2 \quad b = -\frac{23}{6}$
• Add homogeneous and particular solutions to get the general solution.	$f_n = f_n^{(h)} + f_n^{(p)}$ $= a_1 5^n + a_2(-2)^n - 2n - \frac{23}{6}$
• Use initial conditions to set up system.	$n=0: f_0 = \frac{7}{6} = a_1 + a_2 - \frac{23}{6}$ $n=1: f_1 = \frac{-11}{6} = 5a_1 - 2a_2 - \frac{35}{6}$
Solve for $a_1, \dots, a_d$	$a_1 = 2, \quad a_2 = 3$
Plug values for $a_1, \dots, a_d$ back in general solution for f_n to get the final closed form expression for f_n.	$f_n = 2 \cdot 5^n + 3(-2)^n - 2n - \frac{23}{6}$

Thm: The form of particular solutions to certain linear nonhomogeneous recurrence relations

- Suppose the sequence $\{f_n\}$ is described by linear nonhomogeneous RR

$$f_n = c_1(f_{n-1}) + c_2(f_{n-2}) + \dots + c_d(f_{n-d}) + F(n)$$

and suppose that the function $F(n)$ has the form $p(n)\,s^n$, where $p(n)$ is a polynomial of degree t and s is constant.

- If s is not a root of the charactersistic equation for the associated homogeneous RR, then the particular solution has this form:

$$f_n = \left[d_t n^t + (d_{t-1})\, n^{t-1} + \dots + d_1 n + d_0\right] s^n$$

- If s is a root of the charactersistic equation for the associated homogeneous RR of multiplicity m then the particular solution has this form:

$$f_n = n^m \left[d_t n^t + (d_{t-1})\, n^{t-1} + \dots + d_1 n + d_0\right] s^n$$

8.17 Divide-and Conquer RR

The Master Theorem

- Let $a, b,$ and d be constants such that a and b are positive integers with $b \geq 2$, and d is a nonnegative real number.
- If $T(n) = aT(\frac{n}{b}) + \Theta(n^d)$, then:
  - If $\frac{a}{b^d} < 1$, then $T(n) = \Theta(n^d)$
  - If $\frac{a}{b^d} = 1$, then $T(n) = \Theta(n^d \log n)$
  - If $\frac{a}{b^d} > 1$, then $T(n) = \Theta(n^{\log_b a})$

- The theorem holds if $T(n/b)$ is replaced with $T(\lfloor \frac{n}{b} \rfloor)$ or $T(\lceil \frac{n}{b} \rceil)$